International Journal of Theoretical Physics, Vol. 38, No. 10, 1999

Cyclic Representation of the $gl_{\sqrt{q}}(n)$-Covariant Oscillator Algebra

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Received February 12, 1999

The cyclic representation of $gl_{\sqrt{q}}(n)$-covariant oscillator algebra is discussed by use of some noncommutating cyclic variables.

Quantum groups or q-deformed Lie algebras imply some specific deformations of classical Lie algebras. From a mathematical point of view, these are noncommutative associative Hopf algebras. The structure and representation theory of quantum groups were developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4], and Biedenharn [5]. Recently there has been some interest in more general deformations involving arbitrary real functions of weight generators and including q-deformed algebras as a special case [6–10].

Some effort also has been made for the multimode extension of single-mode q-deformed boson algebras. The most successful work was accomplished by Pusz and Woronowicz [11]. They demanded $gl_{\sqrt{q}}(n)$-covariance for the multimode q-boson system. For this reason, I will refer to their algebra as the $gl_{\sqrt{q}}(n)$-covariant multimode oscillator algebra.

In this paper, I discuss the cyclic representation of $gl_{\sqrt{q}}(n)$ algebra with $\sqrt{q}$ a root of unity.

The $gl_{\sqrt{q}}(n)$-covariant multimode oscillator algebra [11] is defined as

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0020-7748/99/1000-266 5$16.00/0

$$
\begin{aligned}
\bar{a}_i\bar{a}_j &= \sqrt{q}\,\bar{a}_j\bar{a}_i \qquad (i < j)\\
a_i a_j &= \frac{1}{\sqrt{q}}\, a_j a_i \; (i < j)\\
a_i\bar{a}_j &= \sqrt{q}\,\bar{a}_j\, a_i \qquad (i \neq j)\\
a_i\bar{a}_i &= 1 + q\bar{a}_i a_i + (q-1)\sum_{k=i+1}^{n}\bar{a}_k a_k\\
[N_i, a_j] &= -\delta_{ij}a_j\\
[N_i, \bar{a}_j] &= \delta_{ij}\bar{a}_j \qquad (i, j = 1, 2, \ldots, n) \qquad (1)
\end{aligned}
$$

When the deformation parameter q is real, $\bar{a}_i$ becomes a hermitian conjugate of a_i. But this is not the case when q is a complex number.

From the algebra (1), one obtains the relation between the number operators and the step operators as follows:

$$\bar{a}_i a_i = q^{\Sigma_{k=i+1}^{n} N_k}[N_i] \qquad (2)$$

where $[x]$ is a q-number and is defined as

$$[x] = \frac{q^x - 1}{q - 1} \qquad (3)$$

Let us introduce the Fock space basis (number basis) $|n_1, n_2, \ldots, n_n\rangle$ for the number operators $N_1, \ldots, N_n$ satisfying

$$N_i|n_1, \ldots, n_n\rangle = n_i|n_1, \ldots, n_n\rangle \qquad (4)$$

where $n_1, \ldots, n_n = 0, 1, \ldots$. For the brevity, we adopt the notation

$$|\mathbf{n}\rangle = |n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + \cdots + n_n\mathbf{e}_n\rangle = |n_1, n_2, \ldots, n_n\rangle$$

Then the matrix representation for the operators N_i, a_i, and $\bar{a}_i$ is given by

$$
\begin{aligned}
N_i|\mathbf{n}\rangle &= n_i|\mathbf{n}\rangle\\
a_i|\mathbf{n}\rangle &= \sqrt{q^{\Sigma_{k=i+1}^{n} n_k}[n_i]}\;|\mathbf{n} - \mathbf{e}_i\rangle\\
\bar{a}_i|\mathbf{n}\rangle &= \sqrt{q^{\Sigma_{k=i+1}^{n} n_k}[n_i + 1]}\;|\mathbf{n} + \mathbf{e}_i\rangle \qquad (5)
\end{aligned}
$$

From Eq. (5) we know that there exists a ground state $|\mathbf{0}\rangle = |0, 0, \ldots, 0\rangle$ satisfying

$$a_i|\mathbf{0}\rangle = 0 \qquad \text{for all} \quad i = 1, 2, \ldots, n$$

Thus the excited state $|\mathbf{n}\rangle$ is obtained by applying the creation operators $(\bar{a}_i)$ to the ground state $|\mathbf{0}\rangle$ successively,

$$|\mathbf{n}\rangle = \frac{\bar{a}_n^{n_n} \cdots \bar{a}_1^{n_1}}{\sqrt{[n_1]! \cdots [n_n]!}} |\mathbf{0}\rangle \tag{6}$$

If we replace

$$k_i = q^{N_i}$$

the fourth equation of Eq. (1) becomes

$$[a_i, \bar{a}_i] = \prod_{l=i}^{n} k_l \tag{7}$$

Now let us introduce the new operator as follows:

$$K_i = \prod_{l=i}^{n} k_l \tag{8}$$

Then the fourth, fifth, and sixth relations of Eq. (1) become

$$\begin{aligned} &[a_i, \bar{a}_i] = K_i \\ &K_i \bar{a}_j K_i^{-1} = \begin{cases} q\bar{a}_j & \text{if} \quad i \le j \\ \bar{a}_j & \text{if} \quad i > j \end{cases} \\ &K_i a_j K_i^{-1} = \begin{cases} q^{-1} a_j & \text{if} \quad i \le j \\ a_j & \text{if} \quad i > j \end{cases} \end{aligned} \tag{9}$$

Now we discuss the cyclic representation of $gl_{\sqrt{q}}(n)$ algebra when the deformation parameter is a root of unity. Let us consider the case that $\sqrt{q}$ is an Nth primitive root of unity,

$$(\sqrt{q})^N = 1$$

Since $gl_{\sqrt{q}}(n)$ algebra describes the n-mode oscillator system, its cyclic representation needs n noncommuting cyclic variables Z_i, $i = 1, 2, \ldots, n$, and another n noncommuting cyclic variables $\mathbf{X}_i$ $i = 1, 2, \ldots, n$. We assume that the commutation relations between these two types of variables are as follows:

$$\begin{aligned} X_i X_j &= \alpha_{ij} X_j X_i \\ Z_i X_j &= \beta_{ij} X_j Z_i \\ Z_i Z_j &= \gamma_{ij} Z_j Z_i \\ X_i^N &= Z_i^N = 1 \end{aligned} \tag{10}$$

From the cyclic representation of single-mode q-boson algebra [12], the cyclic representation of $gl_{\sqrt{q}}(n)$ algebra is assumed to take the following form:

$$\begin{aligned} a_i &= A_i\, Z_n \cdots Z_i\, X_n^{N-1} \cdots X_i^{N-1} \\ \bar{a}_i &= B_i\, X_n \cdots X_i \\ K_i &= Z_i \cdots Z_n \\ K_i^{-1} &= Z_n^{N-1} \cdots Z_i^{N-1} \end{aligned} \tag{11}$$

where A_i and B_i are ordinary compex numbers which will be fixed later.

Since all K_i operators are commuting, Z_i operators should be commuting among them. It means that the coefficients γ_{ij} should be unity. Now we determine the coefficients α_{ij}, β_{ij} so that that realization (11) may satisfy the $gl_{\sqrt{q}}(n)$ algebra.

From the first and second relation of Eq. (1), we have

$$\prod_{n=j}^{n} [\prod_{l=i}^{n} (\beta_{kl}/\beta_{lk}) \prod_{m=i}^{j-1} \alpha_{mk}] = \frac{1}{\sqrt{q}} \tag{12}$$

$$\prod_{k=j}^{n} \prod_{m=i}^{j-1} \alpha_{mk} = \sqrt{q} \tag{13}$$

Equation (13) is easily solved and its simplest solution is

$$\alpha_{ij} = \begin{cases} \sqrt{q} & \text{if} \quad j = i + 1 \\ 1/\sqrt{q} & \text{if} \quad j = 1 - 1 \\ 1 & \text{if} \quad j \neq i \pm 1 \end{cases} \tag{14}$$

From the third relation of Eq. (1) we have

$$\frac{\prod_{t=j}^{i-1} \alpha_{it} \prod_{s=i+1}^{n} \prod_{t=j}^{i-1} \alpha_{st}}{\prod_{k=i}^{n} \prod_{i=j}^{n} \beta_{kl}} = \frac{1}{\sqrt{q}} \qquad (i > j) \tag{15}$$

$$\frac{\prod_{s=i}^{n} \prod_{l=j}^{n} \beta_{kl}}{\prod_{t=j}^{n} \prod_{s=i}^{j-1} \alpha_{st}} = \sqrt{q} \qquad (i < j) \tag{16}$$

Applying Eq. (14) to Eqs. (15) and (16) reduces them to

$$\prod_{k=i}^{n} \prod_{l=j}^{n} \beta_{kl} = \begin{cases} q & \text{if} \quad i < j \\ 1 & \text{if} \quad i > j \end{cases} \tag{17}$$

The simplest solution of Eq. (17) is given by

$$\beta_{ij} = \begin{cases} q & \text{if} \quad i = j \\ q^{-1} & \text{if} \quad j = i - 1 \\ 1 & \text{otherwise} \end{cases} \tag{18}$$

From the first relation of Eq. (19) we have

$$A_i B_i \left(\frac{1}{\sqrt{q}} \right)^{n-i} \left(1 - \frac{1}{\prod_{k=i}^{n} \prod_{l=i}^{n} \beta_{kl}} \right) = 1 \tag{19}$$

Using the solution (18), we find that Eq. (19) reduces to

$$A_i B_i = \frac{(\sqrt{q})^{n-i}}{1 - q^{-1}} \tag{20}$$

These solutions give the cyclic representation of $gl_{\sqrt{q}}(n)$ algebra.

To conclude, I introduced two types of noncommuting cyclic variables (X_i and Z_i) and assumed some commutation relation among them. Using these, I obtained the cyclic representation of $gl_{\sqrt{q}}(n)$ algebra when $\sqrt{q}$ is an Nth primitive root of unity.

ACKNOWLEDGMENT

This paper was supported by the KOSEF (981-0201-003-2).

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